Applications of Combinatorial Geometry to Geometric Optimization

Sathish Govindarajan

Department of Computer Science and Automation
Indian Institute of Science, Bangalore

Expository lectures on Graph and Geometric Algorithms
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Geometric Problems

- $\mathcal{C}$ - Collection of simple geometric objects (rectangles, squares, circles, lines, points)

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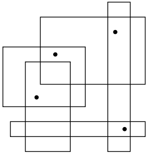

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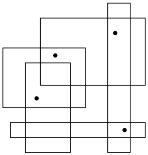

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(Structural, Combinatorial questions)

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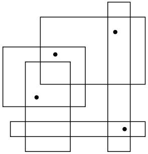

- Combinatorial Geometry: Understand the interactions among objects in $\mathcal{C}$ (Structural, Combinatorial questions)
- Computational Geometry: Design efficient algorithms (Computational questions)

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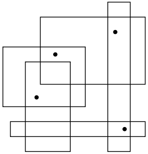


- Combinatorial Geometry: Understand the interactions among objects in $\mathcal{C}$ (Structural, Combinatorial questions)
- Computational Geometry: Design efficient algorithms (Computational questions)
- Connection between these two areas

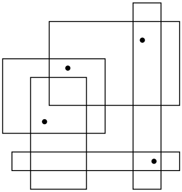
Overview of talk

- Introduction to Combinatorial Geometry
- Approximation algorithms in Geometric Optimization
  - Greedy based
  - Linear Programming based
  - Local Search based

Introduction to Combinatorial Geometry

Combinatorial Geometry

- Structural, Combinatorial properties of a collection of geometric objects



Combinatorial Geometry

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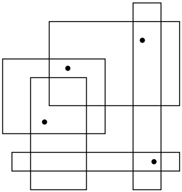

- Sub-areas
  - Geometric graphs, Incidences, Distance based problems, Arrangements, Epsilon nets, Geometric Discrepency
- Classical Theorems
  - Radon's Theorem, Caratheodory Theorem, Tverberg Theorem, Helly's Theorem, Centerpoint Theorem,

Combinatorial Geometry

- Nature of Questions (curious, intuitive, elementary)
- Elegant solutions
- Multiple proofs

Combinatorial Geometry - Illustrative Problem

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A (Curious) Question

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Can we construct a set of points P in the plane such that there is no line that passes through exactly two points of P?

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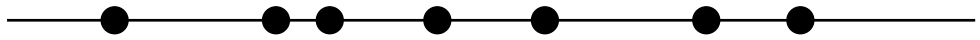
- Yes. All points on a line.



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Revised Question (Curious)

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Can we construct a set of points P in the plane, not all in a line, such that there is no line that passes through exactly two points of P?

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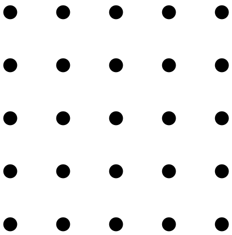
- Yes. Integer grid.



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Theorem (Sylvester-Gallai Theorem)

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- Posed by Sylvester in 1893 and reposed by Erdos in 1943
- Solved by Gallai in 1944
- Many alternate proofs
- Elegant proof by Kelly (Communicated by Coxeter)

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- Many alternate proofs
- Elegant proof by Kelly (Communicated by Coxeter)
- Sylvester-Gallai in finite fields: Connections to showing circuit lower bounds

Combinatorial Geometry - Illustrative Problem

Theorem (Sylvester-Gallai Theorem)

Given any finite set of points P in the plane, not all in a line, there exists a line that passes through exactly two points of P

Proof:

Combinatorial Geometry - Illustrative Problem

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Given any finite set of points P in the plane, not all in a line, there exists a line that passes through exactly two points of P

Proof:

- Look at all the lines $\mathcal{L}$ that connect two points of P
- $d(l)$: distance of closest point from l, $l \in \mathcal{L}$
- L : line in $\mathcal{L}$ with smallest $d(l)$

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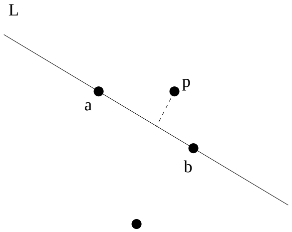
- Claim: L passes through exactly 2 points of P



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Proof (contd):

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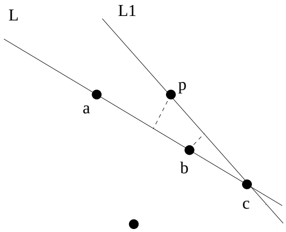
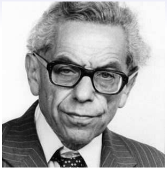
Given any finite set of points P in the plane, not all in a line, there exists a line that passes through exactly two points of P

Proof (contd):

- L : line in $\mathcal{L}$ with smallest $d(l)$
- Claim: L passes through exactly 2 points of P
- Suppose L passes through 3 or more points
- $\exists$ a line $L1 \in \mathcal{L}$ with smaller $d(l)$



Paul Erdos



- Fascination for elegant proofs (Proofs from the “BOOK”)
- 1500 papers
- About 500 co-authors
- Erdos number
- Biography: “My brain is open”

Geometric Optimization using Combinatorial Geometry

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  - Set Cover, Hitting Set, Independent Set, etc

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- Approximation algorithm using Combinatorial Geometry

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- Geometric optimization problems
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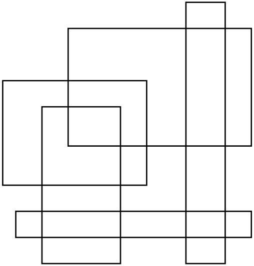
- Approximation algorithm using Combinatorial Geometry
    - Connect the optimization problem to an appropriate combinatorial geometry problem
    - Solve this combinatorial geometry problem
    - Use this to get the approximate solution for the optimization problem

Independent Set

- S - set of m geometric objects
- Compute maximum sized subset $T \subseteq S$ such that all objects in T are "independent", i.e., $r \cap s = \emptyset, \forall r, s \in T$



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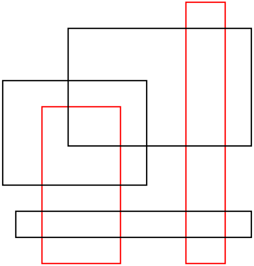

- Motivation
  - Map labelling, data mining, Sensor and wireless networks, Unsplittable flow, . . .

Piercing Set

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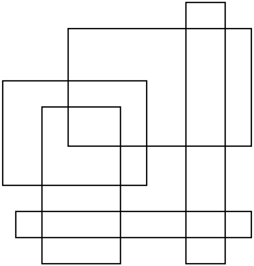
- Compute minimum sized subset $Q \subseteq R^2$ such that all objects in S are "pierced", i.e., $Q \cap r \neq \emptyset, \forall r \in S$



- Motivation
  - Critical facility location, Robotics, Sensor and wireless networks, VLSI, . . .
- Special case of hitting set : $P = R^2$

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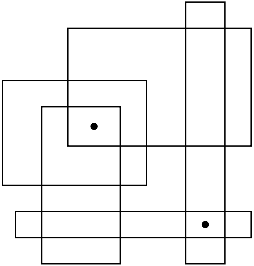

Greedy based

Independent Set of Intervals

- S - set of m intervals on the real line
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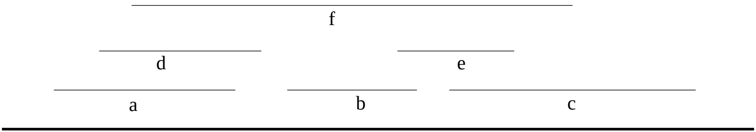

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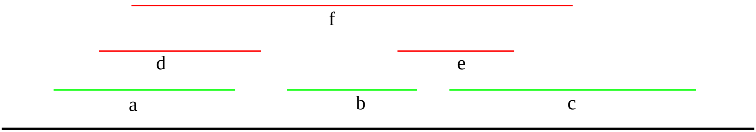

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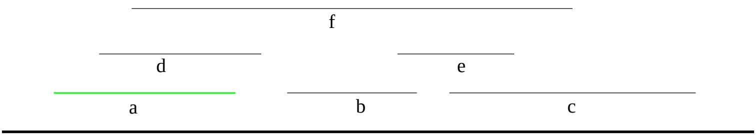

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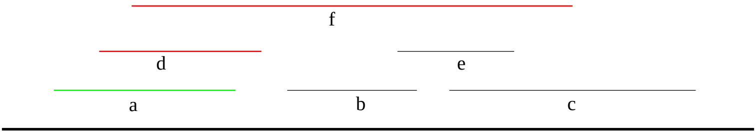

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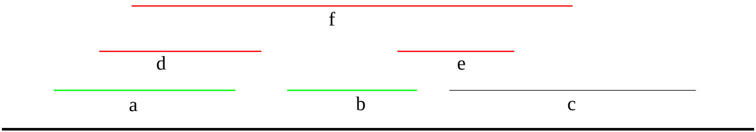

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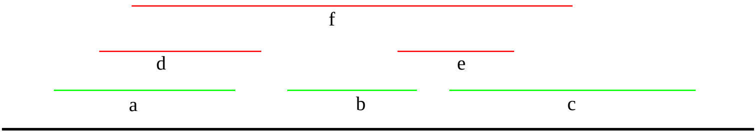

Optimality of Greedy Algorithm

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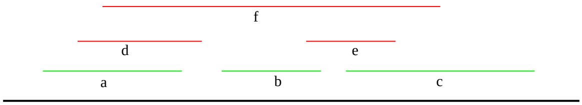

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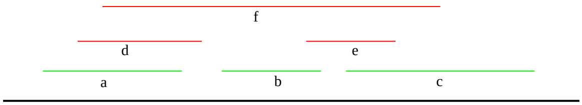

- Why is the Greedy Algorithm optimal?
- Textbook Proof
  - Compare Greedy solution with an optimal solution
  - At each iteration, greedy stays ahead of optimal (using induction)
  - At the end, greedy is not worse than optimal

Combinatorial Problem

- *S* - set of *n* intervals

- ν - Optimal Independent Set size of *S*
- τ - Optimal Piercing Set size of *S*

- Question: Relation between τ and ν
- $\tau \geq \nu$ (Lower bound)
- Question: Upper bound τ as a function of ν (Worst case over all possible *S*)

τ versus ν

- Greedy Algoithm for Independent Set
  - Pick the first ending interval j in I
  - Remove all the intervals that intersect with j
  - Repeat above steps until no intervals are left

τ versus ν

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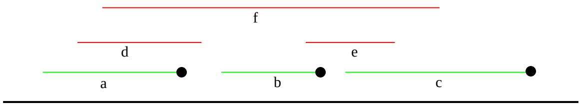
- P is a piercing set for S



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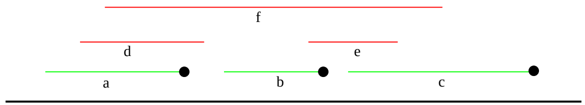

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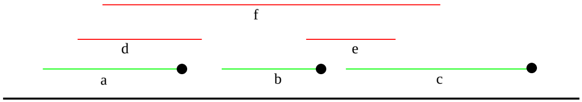

- $\nu \geq |I|$ (ν is optimal independent set size)

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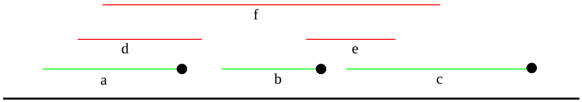

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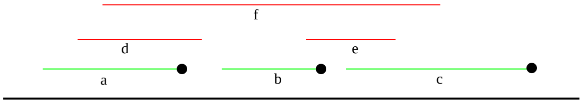

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- $\nu \geq |I| = |P| \geq \tau$ (τ is optimal piercing set size)

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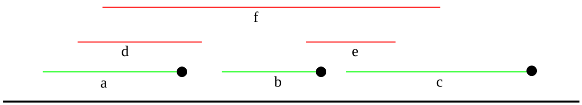

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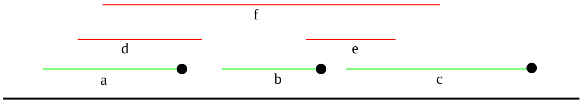

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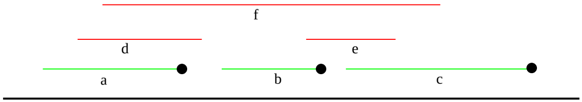

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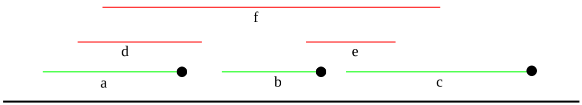

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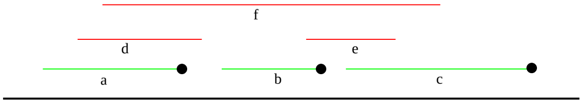

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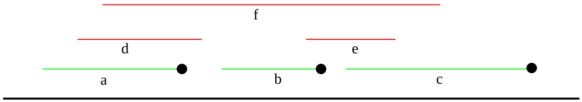

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- $|P| = \tau$ (P is optimal piercing set)
- $\tau = \nu$

Greedy Algo: Independent Set of Squares

- S - set of axis parallel squares in the plane
- Greedy Algoithm

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  - Pick the smallest square s in I
  - Pick the 4 corners of square s in P
  - Remove all the intervals that intersect with square s
  - Repeat above steps until no squares are left

Greedy Algo: Independent Set of Squares

- *S* - set of axis parallel squares in the plane
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  - Pick the smallest square s in I
  - Pick the 4 corners of square s in P
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  - Repeat above steps until no squares are left
- I is an independent set of S

Greedy Algo: Independent Set of Squares

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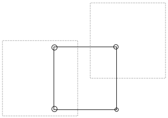

τ versus ν for squares

- I : Greedy Independent Set for squares
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- $|P| = 4 * |I|$

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τ versus ν problem

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- Similar argument for disks (Exercise)

τ versus ν problem

- Similar argument for disks (Exercise)
- Similar argument for fat objects

Linear Programming based Algorithms
Local Search based Algorithms

Hitting Set

- P - set of n points
- S - set of m geometric objects

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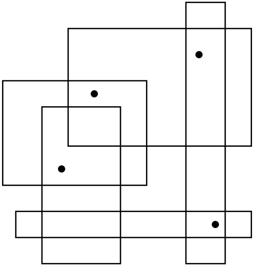

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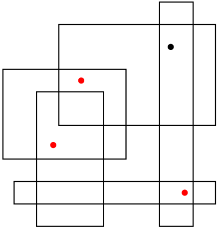

- Motivation
  - Critical facility location, Robotics, Sensor and wireless networks, VLSI, . . .

Hitting Set - Known results

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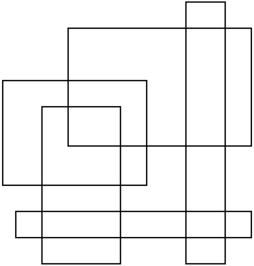

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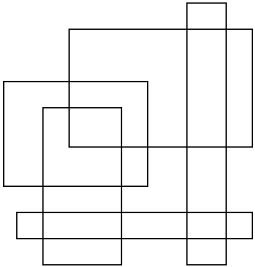

- Special case of hitting set : $P = R^2$

Independent Set

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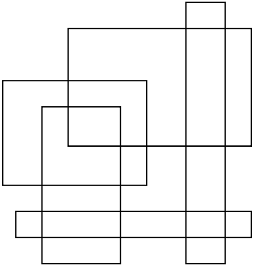

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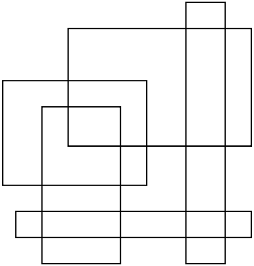

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  - Poly-time solvable for skyline rectangles [CG14]

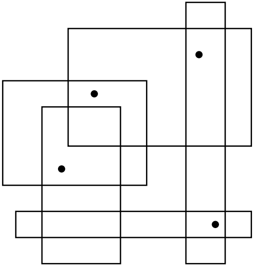
Linear Programming based Approximation

Hitting Set and Set Cover using Epsilon Nets

Hitting Set

- P - set of n points
- S - set of m geometric objects

- Compute minimum sized subset $Q \subseteq P$ such that all objects in S are "hit", i.e., $Q \cap r \neq \emptyset, \forall r \in S$



Linear Programming for Hitting Set

- P - set of n points, S - set of m geometric objects
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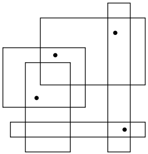

- Indicator variable: x_i for each point $p_i \in P$

$$Minimize \sum_{i=1}^{n} x_i$$

$$\sum_{j, p_j \in O_i} x_j \geq 1 \forall i = 1 \ldots m$$

$$x_i = \{0, 1\} \forall i = 1 \ldots n$$

Combinatorial problem

- Epsilon Nets: Combinatorial problem related to Hitting Set

Epsilon Nets : Formal definition

Definition

Let P be a set of n points in the plane. $N \subset P$ is a ϵ*-net* for a family of geometric objects $\mathcal{S}$ if $S \cap N \neq \emptyset$ for any $S \in \mathcal{S}$ such that $|S \cap P| > \epsilon n$.

- Epsilon nets: Hitting set for **dense** objects

Epsilon Nets

Theorem (ϵ-net theorem (Haussler,Welzl))

Let P be a set of n points and S be a set of geometric objects. Then there exists an ϵ-net of size $O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$.

- Epsilon nets of constant size (independent of n)

Hitting Set using Epsilon Nets

- Epsilon nets: Hitting set for **dense** objects, i.e., each object has $> \epsilon n$ points

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  - ϵ'-net gives a feasible hitting set

Hitting Set using Epsilon Nets

- Finding ϵ' and w_i's

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  - Solving a Linear Program [Evan et al, 2005]
- Suppose Epsilon-nets of size $O(\frac{1}{\epsilon}f(\frac{1}{\epsilon}))$ exists
- Quality of Solution
  - Observation: $\epsilon' \leq \frac{1}{OPT}$
  - Solution size: $O(OPT * f(OPT)$
  - $f(OPT)$ approximation

Epsilon Nets: Results

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- ϵ-nets of size $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ exist for axis parallel rectangles in $\mathbb{R}^2$[Aronov, Ezra, Sharir]
- $\log\log n$-approximation for rectangles [AES09, BG94]

Set Cover using Dual Epsilon Nets

- Set Cover: Dual of Hitting Set
- Set Cover solved using Dual Epsilon Nets [Bronimann, Goodrich '94]
- Set Cover for Disks, Squares
  - Disks, Squares have linear Union Complexity
  - Disks, Squares have linear Dual Epsilon nets [Clarkson, Varadarajan]
  - Set Cover for Disks, Squares have $O(1)$ approximation

Independent Set using Coloring problem

Linear Programming for Independent Set

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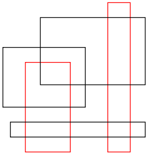
- Compute maximum sized subset $T \subseteq S$ such that all objects in T are "independent", i.e., $r \cap s = \emptyset, \forall r, s \in T$



- P: Place a point in each distinct region
- Indicator variable: x_i for each object $s_i \in S$

$$Maximize \sum_{i=1}^{n} x_i$$

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- Consider the Intersection graph of the geometric objects

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- Function f: $\chi = \omega * f(\omega)$

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- Return the largest color class as the independent set solution
- The coloring bound gives the approximation factor

Coloring problem: Results

- Consider the Intersection graph of the geometric objects
- Coloring problem: Chromatic number as a function of Clique number

- Square, Disks: $\chi = O(\omega)$
- Rectangles(no containment): $\chi = O(\omega \log \omega)$
- Rectangles: $\chi = O(\omega^2)$

Local Search based Approximation

Local Search

- Local Search Paradigm
  - Start with any feasible solution
  - Make improvements to the current solution by making local changes
  - Stop when local improvement is not possible

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- Challenge: Theoretical guarantee on solution quality
- Recently, many local search based approximation algorithms
  - Innovative analysis techniques

Local Search: Geometric Results Overview

- Clustering
  - Facility location [Charikar et al ’05, Cohen et al ’15]
  - k-median [Korupolu et al ’00, Arya et al ’04]
  - k-means [Kanungo et al ’04, Friggstad et al ’16]
- Packing and Covering
  - Hitting Set, Set Cover [Mustafa et al ’09, Govindarajan et al ’16]
  - Independent Set [Chan et al ’09, Ashner et al ’13]
  - Dominating Set [Gibson et al ’10]
  - Terrain guarding [Krohn et al ’14]

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- Stop when none of the $(k, k-1)$-swaps is feasible
- Running time: $O(n^{O(k)})$
  (k is constant)

Analysis Framework [Aschner et al. '13]

- Objective function - Size of the solution

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  - Local exchange property $((A \setminus A') \cup N(A')$ is feasible soln.)

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- $|A| \leq (1+\epsilon)|O|$ (PTAS approximation algorithm)

Local Search for Hitting Set and Independent Set

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- PTAS for Independent Set of Disks, Sqaures [Chan and Har-Peled '09]
  - Ashner Framework Graph: Intersection graph on $A \cup O$ (Planar)

Open Problems - Local Search Analysis

- Weighted Hitting Set and Independent Set
- Unweighted Demand Hitting Set and Demand Set Cover
- Discrete Independent Set

Questions?